A Determinant Formula for a Class of Rational Solutions of Painlevé V Equation

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Abstract

We give an explicit determinant formula for a class of rational solutions of the Painlevé V equation in terms of the universal characters.

1 Introduction and Main Result

It is known that six Painlevé equations are in general irreducible, namely, their solutions cannot be expressed by "classical functions" in the sense of Umemura [18]. However, it is also known that they admit classical solutions for special values of parameters except for $\mathrm{P_I}$. Much effort have been made for the investigation of classical solutions. As a result, it has been recognized that there are two classes of classical solutions. One is transcendental classical solutions expressible in terms of functions of hypergeometric type. Another one is algebraic or rational solutions. It is also known that the Painlevé equations (except for $\mathrm{P_I}$) admit action of the affine Weyl groups as groups of the Bäcklund transformations. It is remarkable that such classical solutions are located on special places from a point of view of symmetry in the parameter spaces [12, 13, 14, 15]. For example, $\mathrm{P_{II}}$, $\mathrm{P_{III}}$ and $\mathrm{P_{IV}}$, whose symmetry is described by the affine Weyl group of type $A_1^{(1)}$, $A_1^{(1)}\times A_1^{(1)}$ and $A_2^{(1)}$, respectively, admit transcendental classical solutions on the reflection hyperplanes, and rational solutions on the barycenters of Weyl chambers of the corresponding affine Weyl group.

Umemura et al have investigated the class of solutions on the barycenters of Weyl chambers and found that (1) these solutions are expressed by some characteristic polynomials generated by the Toda type bilinear equations, (2) the coefficients of such polynomials admit mysterious combinatorial properties [17, 7, 16]. These special polynomials are sometimes referred as *Yablonskii-Vorob'ev polynomials* for $\mathrm{P_{II}}$ [19], *Okamoto polynomials* for $\mathrm{P_{IV}}$ [14], *Umemura polynomials* for $\mathrm{P_{III}}$, $\mathrm{P_V}$ and $\mathrm{P_{VI}}$.

One important aspect among such polynomials is that they are expressed as special cases of the Schur functions. As is well known, the Schur functions are characters of the irreducible polynomial representations of $GL(n)$ and arise as τ-functions of the KP hierarchy [1]. For example, it is known that the special polynomials for $\mathrm{P_{II}}$ and $\mathrm{P_{III}}$ are expressible by 2-reduced Schur functions, and those for $\mathrm{P_{IV}}$ by 3-reduced Schur functions [2, 3, 4, 8].

In this paper, we consider $\mathrm{P_V}$,

$$\frac{d^2y}{dt^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2-\frac{1}{t}\frac{dy}{dt}+\frac{(y-1)^2}{2t^2}\left(\kappa_\infty^2 y-\frac{\kappa_0^2}{y}\right)-(\theta+1)\frac{y}{t}-\frac{y(y+1)}{2(y-1)}, \tag{1.1}$$

with parameters κ_∞, κ_0 and θ, whose symmetry is described by the affine Weyl group $W(A_3^{(1)})$. The aim of this paper is to investigate a class of rational solutions on the barycenters of Weyl chambers and to present an explicit formula for them.

By the analogy from the known cases, it is naively expected that they are expressed in terms of 4-reduced Schur functions. However, our formula is expressed by a generalization of Schur functions, which is called the *universal characters* and defined as follows [6].

Definition 1.1 *Let $p_k = p_k(t^{(1)})$ and $q_k = q_k(t^{(2)})$, $k \in \mathbb{Z}$, be two families of polynomials defined by*

$$\begin{aligned} \sum_{k=0}^{\infty} p_k \eta^k &= \exp\left(\sum_{j=1}^{\infty} t_j^{(1)} \eta^j\right), \quad p_k = 0 \text{ for } k < 0, \\ \sum_{k=0}^{\infty} q_k \eta^k &= \exp\left(\sum_{j=1}^{\infty} t_j^{(2)} \eta^j\right), \quad q_k = 0 \text{ for } k < 0, \end{aligned} \tag{1.2}$$

where $t^{(1)} = (t_1^{(1)}, t_2^{(1)}, \cdots)$ and $t^{(2)} = (t_1^{(2)}, t_2^{(2)}, \cdots)$ are the sets of infinite numbers of variables. For any partitions $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_n^{(1)})$ and $\lambda^{(2)} = (\lambda_1^{(2)}, \lambda_2^{(2)}, \cdots, \lambda_m^{(2)})$, the universal character $S_{\lambda^{(1)},\lambda^{(2)}}(t^{(1)}, t^{(2)})$ is defined as

$$S_{\lambda^{(1)},\lambda^{(2)}}(t^{(1)}, t^{(2)}) = \det {}^t\left(q^-_{\lambda_m^{(2)}}, q^-_{\lambda_{m-1}^{(2)}+1}, \cdots, q^-_{\lambda_1^{(2)}+m-1}, p^+_{\lambda_1^{(1)}-m}, p^+_{\lambda_2^{(1)}-m-1}, \cdots, p^+_{\lambda_n^{(1)}-m-n+1}\right), \tag{1.3}$$

where

$$p_j^+ = {}^t\left(p_j, p_{j+1}, \cdots, p_{j+m+n-1}\right), \quad q_j^- = {}^t\left(q_j, q_{j-1}, \cdots, q_{j-m-n+1}\right). \tag{1.4}$$

Our main result is stated as follows.

Theorem 1.2 *For $m, n \in \mathbb{Z}_{\geq 0}$, we define a family of polynomials $S_{m,n} = S_{m,n}(t, s)$ by specializing $S_{\lambda^{(1)},\lambda^{(2)}}(t^{(1)}, t^{(2)})$ as*

$$\lambda^{(1)} = (n, n-1, \cdots, 2, 1), \quad \lambda^{(2)} = (m, m-1, \cdots, 2, 1), \tag{1.5}$$

$$t_j^{(1)} = -\frac{t}{2} + \frac{2s - m + n}{j}, \quad t_j^{(2)} = \frac{t}{2} + \frac{2s - m + n}{j}, \tag{1.6}$$

where s is a parameter. For $m, n \in \mathbb{Z}_{<0}$, we define $S_{m,n}$ through

$$\begin{aligned} S_{m,n}(t, s) &= (-1)^{m(m+1)/2} S_{-m-1,n}(t, s - m - 1/2), \\ S_{m,n}(t, s) &= (-1)^{n(n+1)/2} S_{m,-n-1}(t, s - n - 1/2). \end{aligned} \tag{1.7}$$

Then,

$$y = -\frac{S_{m,n-1}(t, s) S_{m-1,n}(t, s)}{S_{m-1,n}(t, s-1) S_{m,n-1}(t, s+1)}, \tag{1.8}$$

gives the rational solutions of $P_{\rm V}$ (1.1) with the parameters

$$\kappa_\infty = s, \quad \kappa_0 = s - m + n, \quad \theta = m + n - 1, \tag{1.9}$$

and

$$\kappa_\infty = -s, \quad \kappa_0 = s - m + n, \quad \theta = m + n - 1. \tag{1.10}$$

Similarly,

$$y = \frac{2n+1}{2m+1} \frac{S_{m,n-1}(t, s+1/2) S_{m,n+1}(t, s-1/2)}{S_{m-1,n}(t, s-1/2) S_{m+1,n}(t, s+1/2)}, \tag{1.11}$$

gives the rational solutions of $P_{\rm V}$ (1.1) with the parameters

$$\kappa_\infty = m + 1/2, \quad \kappa_0 = n + 1/2, \quad \theta = 2s - m - n - 1. \tag{1.12}$$

This result covers all the rational solutions obtained by applying the Bäcklund transformations on the particular solution of P_V (1.1),

$$y = -1, \quad \kappa_\infty = s, \quad \kappa_0 = s, \quad \theta = -1. \tag{1.13}$$

Remark. In ref. [5], Kitaev et al gave a complete classification of rational solutions for P_V. Our result covers all the rational solutions of the cases (III) and (IV) in their classification. The first half corresponds to the case (III) and the other does to (IV). Also, Noumi and Yamada presented a determinant formula for a class of rational solutions of P_V in terms of 2-reduced Schur functions [9]. Our result includes their formula as a special case, which is explained in Appendix A.

This paper is organized as follows. In Section 2, we give a brief review for the theory of symmetric form of P_V [8, 10, 11], which provides us with a clear description of symmetry structure and τ-functions for P_V. In Section 3, we construct the rational solutions of P_V by using the theory of symmetric form. Proof of our result is given in Section 4. We mention on the relationship between our result and Yamada's general determinant formula[21] of Jacobi-Trudi type in Section 5.

2 Symmetric Form of Painlevé V Equation

By using the theory of symmetric form for P_V, it is possible to describe the structure of Bäcklund transformations in a unified manner and to construct particular solutions systematically. In this section, we summarize the symmetric form of P_V following refs. [10, 11], and derive bilinear equations satisfied by τ-functions.

2.1 Symmetric Form of P_V

P_V (1.1) is equivalent to the Hamilton system [20]

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad ' = t\frac{d}{dt}, \tag{2.1}$$

with the Hamiltonian

$$H = p(p+t)q(q-1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q-1). \tag{2.2}$$

In fact, putting

$$\kappa_\infty = \alpha_1, \quad \kappa_0 = \alpha_3, \quad \theta = \alpha_2 - \alpha_0 - 1, \tag{2.3}$$

with

$$\alpha_0 = 1 - \alpha_1 - \alpha_2 - \alpha_3, \tag{2.4}$$

we see that equation for $y = 1 - 1/q$ is nothing but P_V (1.1). Setting

$$f_0 = \frac{1}{\sqrt{t}}(t+p), \quad f_1 = \sqrt{t}q, \quad f_2 = -\frac{1}{\sqrt{t}}p, \quad f_3 = \sqrt{t}(1-q), \tag{2.5}$$

we obtain the symmetric form of P_V

$$\begin{aligned} f_0' &= f_0 f_2(f_1 - f_3) + \left(\frac{1}{2} - \alpha_2\right) f_0 + \alpha_0 f_2, \\ f_1' &= f_1 f_3(f_2 - f_0) + \left(\frac{1}{2} - \alpha_3\right) f_1 + \alpha_1 f_3, \\ f_2' &= f_2 f_0(f_3 - f_1) + \left(\frac{1}{2} - \alpha_0\right) f_2 + \alpha_2 f_0, \\ f_3' &= f_3 f_1(f_0 - f_2) + \left(\frac{1}{2} - \alpha_1\right) f_3 + \alpha_3 f_1. \end{aligned} \tag{2.6}$$

We note that the original dependent variable y of $\mathrm{P_V}$ is expressed as

$$y = -\frac{f_3}{f_1}. \tag{2.7}$$

In terms of variables f_i and α_i, the Bäcklund transformations of $\mathrm{P_V}$ are described by a simple form,

$$\begin{aligned}
&s_i(\alpha_i) = -\alpha_i, \quad && s_i(\alpha_j) = \alpha_j + \alpha_i \ (j = i \pm 1), \quad && s_i(\alpha_j) = \alpha_j \ (j \neq i, i \pm 1),\\
&s_i(f_i) = f_i, && s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i} \ (j = i \pm 1), && s_i(f_j) = f_j \ (j \neq i, i \pm 1),\\
&\pi(\alpha_j) = \alpha_{j+1}, && \pi(f_j) = f_{j+1}, &&
\end{aligned} \tag{2.8}$$

where the subscripts $i = 0, 1, 2, 3$ are understood as elements of $\mathbb{Z}/4\mathbb{Z}$. These transformations commute with derivation $'$ and satisfy the fundamental relations

$$\begin{gathered}
s_i^2 = 1, \quad s_i s_j = s_j s_i \ (j \neq i, i \pm 1), \quad s_i s_j s_i = s_j s_i s_j \ (j = i \pm 1),\\
\pi^4 = 1, \quad \pi s_j = s_{j+1} \pi,
\end{gathered} \tag{2.9}$$

which means that transformations s_i $(i = 0, 1, 2, 3)$ generate the affine Weyl group $W(A_3^{(1)})$, and s_i and π generate its extension including the Dynkin diagram automorphisms.

2.2 τ-Functions and Bilinear Equations

In order to obtain simpler transformation properties, we add a correction term which depends only on t to the Hamiltonian (2.2). The corrected Hamiltonian h_0 is introduced as

$$\begin{aligned}
h_0 = f_0 f_1 f_2 f_3 &+ \frac{\alpha_1 + 2\alpha_2 - \alpha_3}{4} f_0 f_1 + \frac{\alpha_1 + 2\alpha_2 + 3\alpha_3}{4} f_1 f_2\\
&- \frac{3\alpha_1 + 2\alpha_2 + \alpha_3}{4} f_2 f_3 + \frac{\alpha_1 - 2\alpha_2 - \alpha_3}{4} f_3 f_0 + \frac{(\alpha_1 + \alpha_3)^2}{4},
\end{aligned} \tag{2.10}$$

and we put $h_j = \pi^j(h_0)$. Then, we have

$$s_i(h_j) = h_j \ (i \neq j), \quad s_i(h_i) = h_i + \sqrt{t}\,\frac{\alpha_i}{f_i}, \quad \pi(h_i) = h_{i+1}. \tag{2.11}$$

We also introduce τ-functions $\tau_i (i = 0, 1, 2, 3)$ by

$$h_i = \frac{\tau_i'}{\tau_i}. \tag{2.12}$$

Then, defining the action of $s_i (i = 0, 1, 2, 3)$ and π on the τ-functions by

$$s_i(\tau_j) = \tau_j \ \ (i \neq j), \quad s_i(\tau_i) = f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i}, \quad \pi(\tau_i) = \tau_{i+1}, \tag{2.13}$$

we see that the fundamental relations (2.9) are preserved, which implies that the Bäcklund transformations can be lifted to the level of τ-functions. It should be remarked that we have from (2.7) and (2.13)

$$y = -\frac{\tau_3 s_3(\tau_3)}{\tau_1 s_1(\tau_1)}. \tag{2.14}$$

By (2.13), the Bäcklund transformations (2.8) are lead to a set of bilinear equations for τ-functions,

$$\begin{aligned}
\tau_0 s_0 s_1(\tau_1) &= s_0(\tau_0)s_1(\tau_1) + \alpha_0\tau_2\tau_3,\\
\tau_1 s_1 s_0(\tau_0) &= s_0(\tau_0)s_1(\tau_1) - \alpha_1\tau_2\tau_3,\\
\tau_1 s_1 s_2(\tau_2) &= s_1(\tau_1)s_2(\tau_2) + \alpha_1\tau_3\tau_0,\\
\tau_2 s_2 s_1(\tau_1) &= s_1(\tau_1)s_2(\tau_2) - \alpha_2\tau_3\tau_0,\\
\tau_2 s_2 s_3(\tau_3) &= s_2(\tau_2)s_3(\tau_3) + \alpha_2\tau_0\tau_1,\\
\tau_3 s_3 s_2(\tau_2) &= s_2(\tau_2)s_3(\tau_3) - \alpha_3\tau_0\tau_1,\\
\tau_3 s_3 s_0(\tau_0) &= s_3(\tau_3)s_0(\tau_0) + \alpha_3\tau_1\tau_2,\\
\tau_0 s_0 s_3(\tau_3) &= s_3(\tau_3)s_0(\tau_0) - \alpha_0\tau_1\tau_2.
\end{aligned} \tag{2.15}$$

Let us define the translation operators T_i $(i = 0, 1, 2, 3)$ by

$$T_1 = \pi s_3 s_2 s_1, \quad T_2 = s_1\pi s_3 s_2, \quad T_3 = s_2 s_1 \pi s_3, \quad T_0 = s_3 s_2 s_1 \pi, \tag{2.16}$$

which commute with each other and satisfy $T_1T_2T_3T_0 = 1$. These operators act on parameters α_i as

$$T_i(\alpha_{i-1}) = \alpha_{i-1} + 1, \quad T_i(\alpha_i) = \alpha_i - 1, \quad T_i(\alpha_j) = \alpha_j \ (j \neq i-1, i), \tag{2.17}$$

and generate the weight lattice of $A_3^{(1)}$. In terms of T_i, τ-functions in (2.15) are expressed as

$$\begin{array}{lll}
\tau_1 = T_1(\tau_0), & \tau_2 = T_1T_2(\tau_0), & \tau_3 = T_0^{-1}(\tau_0),\\
s_0(\tau_0) = T_0^{-1}T_1(\tau_0), & s_1(\tau_1) = T_2(\tau_0), & s_2(\tau_2) = T_1T_3(\tau_0),\\
s_3(\tau_3) = T_3^{-1}(\tau_0), & s_0s_1(\tau_1) = T_1T_2T_0^{-1}(\tau_0), & s_1s_0(\tau_0) = T_2T_0^{-1}(\tau_0),\\
s_1s_2(\tau_2) = T_2T_3(\tau_0), & s_2s_1(\tau_1) = T_3(\tau_0), & s_2s_3(\tau_3) = T_2^{-1}(\tau_0),\\
s_3s_2(\tau_2) = T_1T_0(\tau_0), & s_3s_0(\tau_0) = T_1T_3^{-1}(\tau_0), & s_0s_3(\tau_3) = T_1T_3^{-1}T_0^{-1}(\tau_0).
\end{array} \tag{2.18}$$

Furthermore, we can derive bilinear equations of Toda type.

Proposition 2.1 *We have*

$$\begin{aligned}
T_1(\tau_0)T_1^{-1}(\tau_0) &= \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 + \frac{3\alpha_1 + 2\alpha_2 + \alpha_3}{4}t\right)\tau_0\cdot\tau_0,\\
T_2(\tau_0)T_2^{-1}(\tau_0) &= \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 - \frac{\alpha_1 - 2\alpha_2 - \alpha_3}{4}t\right)\tau_0\cdot\tau_0,\\
T_3(\tau_0)T_3^{-1}(\tau_0) &= \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 - \frac{\alpha_1 + 2\alpha_2 - \alpha_3}{4}t\right)\tau_0\cdot\tau_0,\\
T_0(\tau_0)T_0^{-1}(\tau_0) &= \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 - \frac{\alpha_1 + 2\alpha_2 + 3\alpha_3}{4}t\right)\tau_0\cdot\tau_0,
\end{aligned} \tag{2.19}$$

where D_T is the Hirota's differential operator defined by

$$D_T^m f\cdot g = \left(\frac{d}{dT} - \frac{d}{dT'}\right)^m f(T)g(T')\Big|_{T=T'}, \tag{2.20}$$

and $\dfrac{d}{dT} = t\dfrac{d}{dt}$.

Proof. Using (2.8),(2.13) and (2.16), we have

$$T_1(\tau_0)T_1^{-1}(\tau_0) = [f_1f_2f_3 + (\alpha_1+\alpha_2)f_1 + \alpha_1f_3]\tau_0^2. \tag{2.21}$$

Noticing that

$$h_0' = \sqrt{t}\left(f_1 f_2 f_3 + \frac{\alpha_1 + 2\alpha_2 - \alpha_3}{4} f_1 + \frac{\alpha_1 - 2\alpha_2 - \alpha_1}{4} f_3\right), \tag{2.22}$$

and $f_1 + f_3 = \sqrt{t}$, we get the first equation in (2.19). The other equations are obtained in similar way. ∎

Remark. Bilinear equations (2.15) and (2.19) are overdetermined systems when they are regarded as equations to determine the τ-functions. However, by construction, the consistency of these equations is guaranteed.

2.3 τ-Cocycles

For simplicity, we introduce a notation,

$$\tau_{k,l,m,n} = T_1^k T_2^l T_3^m T_0^n(\tau_0). \tag{2.23}$$

For small k, l, m, n, we observe that $\tau_{k,l,m,n}$ are factorized as

$$\tau_{k,l,m,n} = \phi_{k,l,m,n}\ \tau_0 \left(\frac{\tau_1}{\tau_0}\right)^k \left(\frac{\tau_2}{\tau_1}\right)^l \left(\frac{\tau_3}{\tau_2}\right)^m \left(\frac{\tau_0}{\tau_3}\right)^n, \tag{2.24}$$

where $\phi_{k,l,m,n}$ are some functions of f_i and α_i. Conversely, if we define $\phi_{k,l,m,n}$ by (2.24), it is shown that $\phi_{k,l,m,n}$'s are polynomials in f_i and α_i for any $k, l, m, n \in \mathbb{Z}$ [21]. The functions $\phi_{k,l,m,n}$ are called the τ-cocycles.

It is easy to see from (2.14), (2.18), (2.23) and (2.24) that we have

$$T_1^k T_2^l T_3^m T_0^n(y) = -\frac{\phi_{k,l,m,n-1}\phi_{k,l,m-1,n}}{\phi_{k+1,l,m,n}\phi_{k,l+1,m,n}}, \quad \text{for } k, l, m, n \in \mathbb{Z}. \tag{2.25}$$

Moreover, it follows from (2.13), (2.16), (2.23) and (2.24) that $\phi_{k,l,m,n}$ are determined by the recurrence relations,

$$\begin{aligned}
\phi_{k+1,l,m,n} &= T_1(\phi_{k,l,m,n})[f_2 f_3 f_0 - (\alpha_3 + \alpha_0) f_0 - \alpha_0 f_2]^{k-l} (f_3 f_0 - \alpha_0)^{l-m} f_0^{m-n}, \\
\phi_{k,l+1,m,n} &= T_2(\phi_{k,l,m,n})[f_3 f_0 f_1 - (\alpha_0 + \alpha_1) f_1 - \alpha_1 f_3]^{l-m} (f_0 f_1 - \alpha_1)^{m-n} f_1^{1+n-k}, \\
\phi_{k,l,m+1,n} &= T_3(\phi_{k,l,m,n})[f_0 f_1 f_2 - (\alpha_1 + \alpha_2) f_2 - \alpha_2 f_0]^{m-n} (f_1 f_2 - \alpha_2)^{1+n-k} f_2^{k-l}, \\
\phi_{k,l,m,n+1} &= T_0(\phi_{k,l,m,n})[f_1 f_2 f_3 - (\alpha_2 + \alpha_3) f_3 - \alpha_3 f_1]^{1+n-k} (f_2 f_3 - \alpha_3)^{k-l} f_3^{l-m},
\end{aligned} \tag{2.26}$$

with $\phi_{0,0,0,0} = 1$.

It is possible to write down the bilinear equations for $\phi_{k,l,m,n}$. From (2.18),(2.23) and (2.24), bilinear Bäcklund transformations (2.15) yield to

$$\begin{aligned}
\phi_{k,l,m,n}\phi_{k+1,l+1,m,n-1} &= \phi_{k+1,l,m,n-1}\phi_{k,l+1,m,n} + (\alpha_0 - n + k)\phi_{k+1,l+1,m,n}\phi_{k,l,m,n-1}, \\
\phi_{k+1,l,m,n}\phi_{k,l+1,m,n-1} &= \phi_{k+1,l,m,n-1}\phi_{k,l+1,m,n} - (\alpha_1 - k + l)\phi_{k+1,l+1,m,n}\phi_{k,l,m,n-1}, \\
\phi_{k+1,l,m,n}\phi_{k,l+1,m+1,n} &= \phi_{k,l+1,m,n}\phi_{k+1,l,m+1,n} + (\alpha_1 - k + l)\phi_{k,l,m,n-1}\phi_{k,l,m,n}, \\
\phi_{k+1,l+1,m,n}\phi_{k,l,m+1,n} &= \phi_{k,l+1,m,n}\phi_{k+1,l,m+1,n} - (\alpha_2 - l + m)\phi_{k,l,m,n-1}\phi_{k,l,m,n}, \\
\phi_{k+1,l+1,m,n}\phi_{k,l-1,m,n} &= \phi_{k+1,l,m+1,n}\phi_{k,l,m-1,n} + (\alpha_2 - l + m)\phi_{k,l,m,n}\phi_{k+1,l,m,n}, \\
\phi_{k,l,m,n-1}\phi_{k+1,l,m,n+1} &= \phi_{k+1,l,m+1,n}\phi_{k,l,m-1,n} - (\alpha_3 - m + n)\phi_{k,l,m,n}\phi_{k+1,l,m,n}, \\
\phi_{k,l,m,n-1}\phi_{k+1,l,m-1,n} &= \phi_{k,l,m-1,n}\phi_{k+1,l,m,n-1} + (\alpha_3 - m + n)\phi_{k+1,l,m,n}\phi_{k+1,l+1,m,n}, \\
\phi_{k,l,m,n}\phi_{k+1,l,m-1,n-1} &= \phi_{k,l,m-1,n}\phi_{k+1,l,m,n-1} - (\alpha_0 - n + k)\phi_{k+1,l,m,n}\phi_{k+1,l+1,m,n}.
\end{aligned} \tag{2.27}$$

Similarly, the last two equations in (2.19) are lead to

$$\begin{aligned}
&\phi_{k,l,m+1,n}\phi_{k,l,m-1,n} \\
&\qquad = \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 + \omega_{k,l,m,n} - \frac{\alpha_1 + 2\alpha_2 - \alpha_3 - k - l + 3m - n}{4} t\right)\phi_{k,l,m,n}\cdot\phi_{k,l,m,n}, \\
&\phi_{k,l,m,n+1}\phi_{k,l,m,n-1} \\
&\qquad = \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 + \omega_{k,l,m,n} - \frac{\alpha_1 + 2\alpha_2 + 3\alpha_3 - k - l - m + 3n}{4} t\right)\phi_{k,l,m,n}\cdot\phi_{k,l,m,n},
\end{aligned} \tag{2.28}$$

with

$$\omega_{k,l,m,n} = (\log \tau_0)'' + k\left(\log \frac{\tau_1}{\tau_0}\right)'' + l\left(\log \frac{\tau_2}{\tau_1}\right)'' + m\left(\log \frac{\tau_3}{\tau_2}\right)'' + n\left(\log \frac{\tau_0}{\tau_3}\right)''. \tag{2.29}$$

3 Construction of Rational Solutions

In this section, we construct the rational solutions of $\mathrm{P_V}$ by using the results in the previous section.

It is obvious that the symmetric form of $\mathrm{P_V}$ (2.6) with (2.4) has a solution,

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(\frac{1}{2} - s, s, \frac{1}{2} - s, s\right), \quad f_i = \frac{\sqrt{t}}{2} \text{ for } i = 0, 1, 2, 3, \tag{3.1}$$

which is on the fixed points with respect to the transformation π^2 and is equivalent to the following solution of $\mathrm{P_V}$,

$$y = -1, \quad \kappa_\infty = s, \quad \kappa_0 = s, \quad \theta = -1. \tag{3.2}$$

This is the unique rational solution in the fundamental region of the affine Weyl group $W(A_3^{(1)})$ in the parameter space, except for the special cases of transcendental classical solutions [20]. Applying Bäcklund transformations to the seed solution (3.1), we obtain the family of rational solutions of $\mathrm{P_V}$. Note that we have

$$T_2^l T_0^l(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(\frac{1}{2} - \tilde{s}, \tilde{s}, \frac{1}{2} - \tilde{s}, \tilde{s}\right), \quad \tilde{s} = s + l, \quad l \in \mathbb{Z}, \tag{3.3}$$

under the specialization (3.1). Comparing (3.3) with (3.1), we see that the effect of T_2 is absorbed by that of T_0^{-1} and shift of the parameter s. Then, we do not need to consider the Bäcklund transformation T_2 for constructing the family of rational solutions of $\mathrm{P_V}$. Taking the initial condition (3.1) into account, we consider the bilinear Bäcklund transformations in terms of the τ-cocycles (2.27). Denoting $\phi_{0,l,m,n} = \phi_{l,m,n}$ in view of the relation $T_1T_2T_3T_0 = 1$, it is easy to derive the following.

Lemma 3.1 *Under the specialization (3.1), bilinear Bäcklund transformations (2.27) are reduced to*

$$\begin{aligned}
&\phi_{0,m,n}\phi_{0,m-1,n-2} = \phi_{-1,m-1,n-2}\phi_{1,m,n} + (1/2 - s - n)\phi_{0,m-1,n-1}\phi_{0,m,n-1},\\
&\phi_{-1,m-1,n-1}\phi_{1,m,n-1} = \phi_{-1,m-1,n-2}\phi_{1,m,n} - s\phi_{0,m-1,n-1}\phi_{0,m,n-1},\\
&\phi_{-1,m-1,n-1}\phi_{1,m+1,n} = \phi_{1,m,n}\phi_{-1,m,n-1} + s\phi_{0,m,n-1}\phi_{0,m,n},\\
&\phi_{0,m-1,n-1}\phi_{0,m+1,n} = \phi_{1,m,n}\phi_{-1,m,n-1} - (1/2 - s + m)\phi_{0,m,n-1}\phi_{0,m,n},\\
&\phi_{0,m-1,n-1}\phi_{-1,m,n} = \phi_{-1,m,n-1}\phi_{0,m-1,n} + (1/2 - s + m)\phi_{0,m,n}\phi_{-1,m-1,n-1},\\
&\phi_{0,m,n-1}\phi_{-1,m-1,n} = \phi_{-1,m,n-1}\phi_{0,m-1,n} - (s - m + n)\phi_{0,m,n}\phi_{-1,m-1,n-1},\\
&\phi_{0,m,n-1}\phi_{-1,m-2,n-1} = \phi_{0,m-1,n}\phi_{-1,m-1,n-2} + (s - m + n)\phi_{-1,m-1,n-1}\phi_{0,m-1,n-1},\\
&\phi_{0,m,n}\phi_{-1,m-2,n-2} = \phi_{0,m-1,n}\phi_{-1,m-1,n-2} - (1/2 - s - n)\phi_{-1,m-1,n-1}\phi_{0,m-1,n-1}.
\end{aligned} \tag{3.4}$$

Moreover, from (2.25), the function

$$y = -\frac{\phi_{0,m,n-1}\phi_{0,m-1,n}}{\phi_{-1,m-1,n-1}\phi_{1,m,n}}, \tag{3.5}$$

solves P_V (1.1) with parameters

$$\kappa_\infty = s, \quad \kappa_0 = s - m + n, \quad \theta = m + n - 1. \tag{3.6}$$

From the recurrence relations (2.26), we observe that $\phi_{l,m,n}$ for small l, m, n are expressed as

$$\phi_{l,m,n} = \left(\frac{\sqrt{t}}{2}\right)^{(m-n-l-1)(m-n-l)/2} U_{l,m,n}, \tag{3.7}$$

where $U_{l,m,n}$ are some polynomials in t and s. Therefore, we next rewrite (3.4) in terms of U. The polynomials $U_{l,m,n}$ have symmetry described by the following lemma.

Lemma 3.2 *The polynomials $U_{l,m,n}$ defined by (3.7) satisfy*

$$U_{1,m,n}(t,s) = U_{0,m,n-1}(t,s+1), \quad U_{-1,m,n}(t,s) = U_{0,m,n+1}(t,s-1). \tag{3.8}$$

Proof. Lemma 3.2 is proved by considering the Toda type equation for τ-cocycles. Under the specialization (3.1), the Hamiltonians and τ-functions are calculated as

$$h_0 = h_2 = \frac{t^2}{16} + s^2, \quad h_1 = h_3 = \frac{t^2}{16} + \left(\frac{1}{2} - s\right)^2, \tag{3.9}$$

and

$$\tau_0 = \tau_2 = t^{s^2} \exp\left(\frac{t^2}{32}\right), \quad \tau_1 = \tau_3 = t^{(1/2-s)^2} \exp\left(\frac{t^2}{32}\right), \tag{3.10}$$

up to the multiplication by some constants, respectively. Then, Toda type bilinear equations (2.28) yield to

$$\begin{aligned} \phi_{l,m+1,n}\phi_{l,m-1,n} &= \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 + \frac{t^2}{8} - \frac{-2s+1-l+3m-n}{4}t\right)\phi_{l,m,n}\cdot\phi_{l,m,n}, \\ \phi_{l,m,n+1}\phi_{l,m,n-1} &= \frac{1}{\sqrt{t}}\left(\frac{1}{2}D_T^2 + \frac{t^2}{8} - \frac{2s+1-l-m+3n}{4}t\right)\phi_{l,m,n}\cdot\phi_{l,m,n}. \end{aligned} \tag{3.11}$$

Substituting (3.7) into (3.11), we obtain Toda type bilinear equations to be satisfied by $U_{m,n} = U_{0,m,n}(t,s)$

$$\begin{aligned} &U_{m+1,n}U_{m-1,n} = \\ &\quad 2t\left[\left(\frac{d^2U_{m,n}}{dt^2}\right)U_{m,n} - \left(\frac{dU_{m,n}}{dt}\right)^2\right] + 2\frac{dU_{m,n}}{dt}U_{m,n} + \left(\frac{t}{4} - \frac{-2s+1+3m-n}{2}\right)U_{m,n}^2, \\ &U_{m,n+1}U_{m,n-1} = \\ &\quad 2t\left[\left(\frac{d^2U_{m,n}}{dt^2}\right)U_{m,n} - \left(\frac{dU_{m,n}}{dt}\right)^2\right] + 2\frac{dU_{m,n}}{dt}U_{m,n} + \left(\frac{t}{4} - \frac{2s+1-m+3n}{2}\right)U_{m,n}^2, \end{aligned} \tag{3.12}$$

with initial conditions

$$U_{-1,-1} = U_{-1,0} = U_{0,-1} = U_{0,0} = 1. \tag{3.13}$$

The functions $U_{m,n} = U_{m,n}(t,s)$ are uniquely determined by Toda equations (3.12) from the initial conditions (3.13) for any $m,n \in \mathbb{Z}$. Moreover, we see that $U_{\pm1,m,n}(t,s)$ satisfy the same Toda equations as $U_{0,m,n\mp1}(t,s\pm1)$, respectively, by the similar calculation. Since the initial conditions for $U_{1,m,n}$ and $U_{-1,m,n}$ are given by

$$\begin{aligned} U_{1,-1,0} = U_{1,0,0} = U_{1,-1,1} = U_{1,0,1} = 1, \\ U_{-1,-1,-2} = U_{-1,-1,-1} = U_{-1,0,-2} = U_{-1,0,-1} = 1, \end{aligned} \tag{3.14}$$

the lemma is proved. ∎

From Lemma 3.2, bilinear Bäcklund transformations (3.4) are rewritten in terms of U.

Proposition 3.3 *Let $U_{m,n} = U_{m,n}(t,s)$ $(m,n \in \mathbb{Z})$ be polynomials which satisfy the bilinear equations,*

$$\begin{aligned} 4U_{m,n+1}U_{m-1,n-1} &= tU^-_{m-1,n}U^+_{m,n} - 2(2s+2n+1)U_{m-1,n}U_{m,n}, \\ 4U^-_{m-1,n+1}U^+_{m,n-1} &= tU^-_{m-1,n}U^+_{m,n} - 4sU_{m-1,n}U_{m,n}, \\ 4U^-_{m-1,n}U^+_{m+1,n-1} &= tU^+_{m,n-1}U^-_{m,n} + 4sU_{m,n-1}U_{m,n}, \\ 4U_{m-1,n-1}U_{m+1,n} &= tU^+_{m,n-1}U^-_{m,n} + 2(2s-2m-1)U_{m,n-1}U_{m,n}, \\ 4U_{m-1,n-1}U^-_{m,n+1} &= tU^-_{m,n}U_{m-1,n} - 2(2s-2m-1)U_{m,n}U^-_{m-1,n}, \\ 4U_{m,n-1}U^-_{m-1,n+1} &= tU^-_{m,n}U_{m-1,n} - 4(s-m+n)U_{m,n}U^-_{m-1,n}, \\ 4U_{m+1,n-1}U^-_{m-1,n} &= tU_{m,n}U^-_{m,n-1} + 4(s-m+n-1)U^-_{m,n}U_{m,n-1}, \\ 4U_{m+1,n}U^-_{m-1,n-1} &= tU_{m,n}U^-_{m,n-1} + 2(2s+2n-1)U^-_{m,n}U_{m,n-1}, \end{aligned} \tag{3.15}$$

with

$$U_{-1,-1} = U_{-1,0} = U_{0,-1} = U_{0,0} = 1, \tag{3.16}$$

where we denote $U^{\pm}_{m,n} = U_{m,n}(t, s \pm 1)$. *Then,*

$$y = -\frac{U_{m,n-1}(t,s)U_{m-1,n}(t,s)}{U_{m-1,n}(t,s-1)U_{m,n-1}(t,s+1)}, \tag{3.17}$$

gives the rational solutions of P_{V} *(1.1) with parameters*

$$\kappa_\infty = s, \quad \kappa_0 = s - m + n, \quad \theta = m + n - 1. \tag{3.18}$$

4 Proof of Theorem 1.2

In this section, we give the proof for Theorem 1.2.

Definition 4.1 *Let* $p_k^{(r)} = p_k^{(r)}(x)$ *and* $q_k^{(r)} = q_k^{(r)}(x)$ *be polynomials defined by*

$$\sum_{k=0}^{\infty} p_k^{(r)} \eta^k = (1-\eta)^{-r} \exp\left(-\frac{x\eta}{1-\eta}\right), \quad p_k^{(r)} = 0 \text{ for } k < 0, \tag{4.1}$$

$$q_k^{(r)}(x) = p_k^{(r)}(-x), \tag{4.2}$$

respectively. For $m, n \in \mathbb{Z}_{\geq 0}$, *we define a family of polynomials* $R^{(r)}_{m,n} = R^{(r)}_{m,n}(x)$ *by*

$$R^{(r)}_{m,n}(x) = \begin{vmatrix} q_1^{(r)} & q_0^{(r)} & \cdots & q_{-m+2}^{(r)} & q_{-m+1}^{(r)} & \cdots & q_{-m-n+3}^{(r)} & q_{-m-n+2}^{(r)} \\ q_3^{(r)} & q_2^{(r)} & \cdots & q_{-m+4}^{(r)} & q_{-m+3}^{(r)} & \cdots & q_{-m-n+5}^{(r)} & q_{-m-n+4}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1}^{(r)} & q_{2m-2}^{(r)} & \cdots & q_m^{(r)} & q_{m-1}^{(r)} & \cdots & q_{m-n+1}^{(r)} & q_{m-n}^{(r)} \\ p_{n-m}^{(r)} & p_{n-m+1}^{(r)} & \cdots & p_{n-1}^{(r)} & p_n^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+4}^{(r)} & p_{-n-m+5}^{(r)} & \cdots & p_{-n+3}^{(r)} & p_{-n+4}^{(r)} & \cdots & p_2^{(r)} & p_3^{(r)} \\ p_{-n-m+2}^{(r)} & p_{-n-m+3}^{(r)} & \cdots & p_{-n+1}^{(r)} & p_{-n+2}^{(r)} & \cdots & p_0^{(r)} & p_1^{(r)} \end{vmatrix}. \tag{4.3}$$

For $m, n \in \mathbb{Z}_{<0}$, *we define* $R^{(r)}_{m,n}$ *through*

$$R^{(r)}_{m,n} = (-1)^{m(m+1)/2} R^{(r)}_{-m-1,n}, \quad R^{(r)}_{m,n} = (-1)^{n(n+1)/2} R^{(r)}_{m,-n-1}. \tag{4.4}$$

Remark. The polynomials p_k and q_k ($k \geq 0$) are essentially the Laguerre polynomials, namely, $p_k^{(r)}(x) = L_k^{(r-1)}(x)$. Moreover, $R^{(r)}_{m,n}$ is related to $S_{m,n}$ in Theorem 1.2 as

$$R^{(r)}_{m,n}(x) = S_{m,n}(t,s), \quad x = \frac{t}{2}, \quad r = 2s - m + n. \tag{4.5}$$

Proposition 4.2 *For $m, n \in \mathbb{Z}$, $R_{m,n}^{(r)}$ satisfy the following bilinear equations.*

$$\begin{aligned}
-(2n+1)R_{m,n+1}^{(r+1)}R_{m-1,n-1}^{(r)} &= xR_{m-1,n}^{(r-1)}R_{m,n}^{(r+2)} - (r+m+n+1)R_{m-1,n}^{(r+1)}R_{m,n}^{(r)},\\
-(2n+1)R_{m-1,n+1}^{(r)}R_{m,n-1}^{(r+1)} &= xR_{m-1,n}^{(r-1)}R_{m,n}^{(r+2)} - (r+m-n)R_{m-1,n}^{(r+1)}R_{m,n}^{(r)},\\
(2m+1)R_{m-1,n}^{(r-1)}R_{m+1,n-1}^{(r)} &= xR_{m,n-1}^{(r+1)}R_{m,n}^{(r-2)} + (r+m-n)R_{m,n-1}^{(r-1)}R_{m,n}^{(r)},\\
(2m+1)R_{m-1,n-1}^{(r)}R_{m+1,n}^{(r-1)} &= xR_{m,n-1}^{(r+1)}R_{m,n}^{(r-2)} + (r-m-n-1)R_{m,n-1}^{(r-1)}R_{m,n}^{(r)},\\
-(2n+1)R_{m-1,n-1}^{(r)}R_{m,n+1}^{(r-1)} &= xR_{m,n}^{(r-2)}R_{m-1,n}^{(r+1)} - (r-m-n-1)R_{m,n}^{(r)}R_{m-1,n}^{(r-1)},\\
-(2n+1)R_{m,n-1}^{(r-1)}R_{m-1,n+1}^{(r)} &= xR_{m,n}^{(r-2)}R_{m-1,n}^{(r+1)} - (r-m+n)R_{m,n}^{(r)}R_{m-1,n}^{(r-1)},\\
(2m+1)R_{m+1,n-1}^{(r-2)}R_{m-1,n}^{(r-1)} &= xR_{m,n}^{(r)}R_{m,n-1}^{(r-3)} + (r-m+n-2)R_{m,n}^{(r-2)}R_{m,n-1}^{(r-1)},\\
(2m+1)R_{m+1,n}^{(r-1)}R_{m-1,n-1}^{(r-2)} &= xR_{m,n}^{(r)}R_{m,n-1}^{(r-3)} + (r+m+n-1)R_{m,n}^{(r-2)}R_{m,n-1}^{(r-1)}.
\end{aligned} \tag{4.6}$$

Comparing (4.6) with (3.15), we obtain an explicit formula for $U_{m,n} = U_{m,n}(t,s)$ in Proposition 3.3.

Proposition 4.3 *We have*

$$U_{m,n}(t,s) = c_m d_n S_{m,n}(t,s), \quad m, n \in \mathbb{Z}, \tag{4.7}$$

where c_m and d_n are constants determined by

$$\begin{aligned}
c_{m+1}c_{m-1} &= \left(m+\frac{1}{2}\right)c_m^2, \quad c_{-1} = c_0 = 1,\\
d_{n+1}d_{n-1} &= -\left(n+\frac{1}{2}\right)d_n^2, \quad d_{-1} = d_0 = 1.
\end{aligned} \tag{4.8}$$

Proof. Putting

$$U_{m,n}(t,s) = c_m d_n R_{m,n}^{(r)}(x), \tag{4.9}$$

with

$$x = \frac{t}{2}, \quad r = 2s - m + n, \tag{4.10}$$

we find that the bilinear relations (4.6) become (3.15). Taking (4.5) into account, we obtain Proposition 4.3. ∎

Applying s_1 to the solutions (1.8) with (1.9), we get the solutions (1.8) with (1.10). Then, the first half of Theorem 1.2 is a direct consequence of Proposition 3.3 and 4.3. It is easy to find that the latter half of Theorem 1.2 is obtained by applying πs_1 to the solutions (1.8) with (1.9). Therefore, now the proof of Theorem 1.2 is reduced to that of Proposition 4.2.

It is possible to reduce the number of bilinear equations to be proved in (4.6) by the following symmetry of $R_{m,n}^{(r)}(x)$.

Lemma 4.4 *We have the relations for $m, n \in \mathbb{Z}_{\geq 0}$*

$$R_{n,m}^{(r)}(-x) = R_{m,n}^{(r)}(x), \tag{4.11}$$

$$R_{n,m}^{(-r)}(x) = (-1)^{m(m+1)/2+n(n+1)/2}R_{m,n}^{(r)}(x). \tag{4.12}$$

Proof. The first relation (4.11) is easily obtained from (4.3). To verify the second relation (4.12), we introduce polynomials $\bar{q}_k^{(r)} = \bar{q}_k^{(r)}(x)$ by

$$\sum_{k=0}^{\infty} \bar{q}_k^{(r)}\eta^k = (1+\eta)^r \exp\left(\frac{x\eta}{1+\eta}\right), \quad \bar{q}_k^{(r)} = 0 \text{ for } k < 0. \tag{4.13}$$

Comparing the generating function for q_k with that for $\bar{q}_k$, we see that each $\bar{q}_k^{(r)}(x)$ is a linear combination of $q_j^{(r)}(x),\ j=k,k-2,k-4,\cdots$. Therefore we can express $R_{m,n}^{(r)}$ for $m,n\in\mathbb{Z}_{\geq 0}$ in terms of p_k and $\bar{q}_k$ as

$$R_{m,n}^{(r)}(x)=\begin{vmatrix} \bar{q}_1^{(r)} & \bar{q}_0^{(r)} & \cdots & \bar{q}_{-m+2}^{(r)} & \bar{q}_{-m+1}^{(r)} & \cdots & \bar{q}_{-m-n+3}^{(r)} & \bar{q}_{-m-n+2}^{(r)} \\ \bar{q}_3^{(r)} & \bar{q}_2^{(r)} & \cdots & \bar{q}_{-m+4}^{(r)} & \bar{q}_{-m+3}^{(r)} & \cdots & \bar{q}_{-m-n+5}^{(r)} & \bar{q}_{-m-n+4}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{q}_{2m-1}^{(r)} & \bar{q}_{2m-2}^{(r)} & \cdots & \bar{q}_{m}^{(r)} & \bar{q}_{m-1}^{(r)} & \cdots & \bar{q}_{m-n+1}^{(r)} & \bar{q}_{m-n}^{(r)} \\ p_{n-m}^{(r)} & p_{n-m+1}^{(r)} & \cdots & p_{n-1}^{(r)} & p_{n}^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+4}^{(r)} & p_{-n-m+5}^{(r)} & \cdots & p_{-n+3}^{(r)} & p_{-n+4}^{(r)} & \cdots & p_{2}^{(r)} & p_{3}^{(r)} \\ p_{-n-m+2}^{(r)} & p_{-n-m+3}^{(r)} & \cdots & p_{-n+1}^{(r)} & p_{-n+2}^{(r)} & \cdots & p_{0}^{(r)} & p_{1}^{(r)} \end{vmatrix}. \tag{4.14}$$

Noticing that $\bar{q}_k$ and p_k are related as

$$\bar{q}_k^{(r)}(x)=(-1)^k p_k^{(-r)}(x), \tag{4.15}$$

we obtain the relation (4.12). ∎

From the symmetries of $R_{m,n}^{(r)}(x)$ described by (4.4) and Lemma 4.4, it is sufficient to prove the first two equations in (4.6) for $m,n\in\mathbb{Z}_{\geq 0}$, which are equivalent to

$$R_{m-1,n+1}^{(r)}R_{m,n-1}^{(r+1)}-R_{m,n+1}^{(r+1)}R_{m-1,n-1}^{(r)}-R_{m-1,n}^{(r+1)}R_{m,n}^{(r)}=0, \tag{4.16}$$

$$-(2n+1)R_{m-1,n+1}^{(r)}R_{m,n-1}^{(r+1)}=xR_{m-1,n}^{(r-1)}R_{m,n}^{(r+2)}-(r+m-n)R_{m-1,n}^{(r+1)}R_{m,n}^{(r)}. \tag{4.17}$$

In the following, we show that these bilinear equations are reduced to Jacobi's identity of determinants. Let D be an $(m+n+1)\times(m+n+1)$ determinant and $D\begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k\end{bmatrix}$ the minor which are obtained by deleting the rows with indices $i_1,\cdots,i_k$ and the columns with indices $j_1,\cdots,j_k$. Then we have Jacobi's identity

$$D\cdot D\begin{bmatrix} m & m+1 \\ 1 & m+n+1\end{bmatrix}=D\begin{bmatrix} m \\ 1\end{bmatrix}D\begin{bmatrix} m+1 \\ m+n+1\end{bmatrix}-D\begin{bmatrix} m+1 \\ 1\end{bmatrix}D\begin{bmatrix} m \\ m+n+1\end{bmatrix}. \tag{4.18}$$

We first choose proper determinants as D (D itself should be expressed in terms of $R_{m,n}^{(r)}$). Secondly, we construct such formulas that express the minor determinants by $R_{m,n}^{(r)}$. Then, Jacobi's identity yields bilinear equations for $R_{m,n}^{(r)}$ which are nothing but (4.16) and (4.17).

We have the following lemmas.

Lemma 4.5 *We put*

$$D\equiv\begin{vmatrix} -q_1^{(r+1)} & q_1^{(r)} & \cdots & q_{-m-n+3}^{(r)} & q_{-m-n+2}^{(r)} \\ -q_3^{(r+1)} & q_3^{(r)} & \cdots & q_{-m-n+5}^{(r)} & q_{-m-n+4}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -q_{2m-1}^{(r+1)} & q_{2m-1}^{(r)} & \cdots & q_{m-n+1}^{(r)} & q_{m-n}^{(r)} \\ p_{n-m+1}^{(r+1)} & p_{n-m+2}^{(r)} & \cdots & p_{2n}^{(r)} & p_{2n+1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+3}^{(r+1)} & p_{-n-m+4}^{(r)} & \cdots & p_{2}^{(r)} & p_{3}^{(r)} \\ p_{-n-m+1}^{(r+1)} & p_{-n-m+2}^{(r)} & \cdots & p_{0}^{(r)} & p_{1}^{(r)} \end{vmatrix}. \tag{4.19}$$

Then, we have

$$
\begin{aligned}
&D=(-1)^m R^{(r+1)}_{m,n+1},\quad D\begin{bmatrix} m\\ 1\end{bmatrix}=R^{(r)}_{m-1,n+1},\\
&D\begin{bmatrix} m+1\\ 1\end{bmatrix}=R^{(r)}_{m,n},\quad D\begin{bmatrix} m\\ m+n+1\end{bmatrix}=(-1)^{m-1}R^{(r+1)}_{m-1,n},\\
&D\begin{bmatrix} m+1\\ m+n+1\end{bmatrix}=(-1)^m R^{(r+1)}_{m,n-1},\quad D\begin{bmatrix} m & m+1\\ 1 & m+n+1\end{bmatrix}=R^{(r)}_{m-1,n-1}.
\end{aligned}
\tag{4.20}
$$

Lemma 4.6 *We put*

$$
D\equiv\begin{vmatrix}
\tilde{q}_1^{(r-m-n+2)} & q_1^{(r-m-n+1)} & q_0^{(r-m-n+2)} & \cdots & q^{(r)}_{-m-n+2}\\
\tilde{q}_3^{(r-m-n+2)} & q_3^{(r-m-n+1)} & q_2^{(r-m-n+2)} & \cdots & q^{(r)}_{-m-n+4}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\tilde{q}_{2m-1}^{(r-m-n+2)} & q_{2m-1}^{(r-m-n+1)} & q_{2m-2}^{(r-m-n+2)} & \cdots & q^{(r)}_{m-n}\\
(-1)^{m+n}\hat{p}_{2n}^{(r-m-n+2)} & (-1)^{m+n}p_{2n+1}^{(r-m-n+1)} & (-1)^{m+n-1}p_{2n+1}^{(r-m-n+2)} & \cdots & (-1)^1 p^{(r)}_{2n+1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
(-1)^{m+n}\hat{p}_{2}^{(r-m-n+2)} & (-1)^{m+n}p_{3}^{(r-m-n+1)} & (-1)^{m+n-1}p_{3}^{(r-m-n+2)} & \cdots & (-1)^1 p^{(r)}_{3}\\
(-1)^{m+n}\hat{p}_{0}^{(r-m-n+2)} & (-1)^{m+n}p_{1}^{(r-m-n+1)} & (-1)^{m+n-1}p_{1}^{(r-m-n+2)} & \cdots & (-1)^1 p^{(r)}_{1}
\end{vmatrix},
\tag{4.21}
$$

where $\hat{p}^{(r)}_{2k}$ *and* $\tilde{q}^{(r)}_{2k-1}$ *are defined by*

$$
\hat{p}^{(r)}_{2k}=\frac{p^{(r)}_{2k}}{2k+1},\quad \tilde{q}^{(r)}_{2k-1}=\frac{q^{(r)}_{2k-1}}{r+2k-2}.
\tag{4.22}
$$

Then, we have

$$
\begin{aligned}
&D=\frac{x^{m+n}}{\displaystyle\prod_{j=0}^{n}(2j+1)\prod_{k=1}^{m}(r-m-n+2k)}R^{(r+2)}_{m,n},\\
&D\begin{bmatrix} m\\ 1\end{bmatrix}=(-1)^{n+1}R^{(r)}_{m-1,n+1},\quad D\begin{bmatrix} m+1\\ 1\end{bmatrix}=(-1)^n R^{(r)}_{m,n},\\
&D\begin{bmatrix} m\\ m+n+1\end{bmatrix}=(-1)^{n+1}\frac{x^{m+n-1}}{\displaystyle\prod_{j=0}^{n}(2j+1)\prod_{k=1}^{m-1}(r-m-n+2k)}R^{(r+1)}_{m-1,n},\\
&D\begin{bmatrix} m+1\\ m+n+1\end{bmatrix}=(-1)^{n}\frac{x^{m+n-1}}{\displaystyle\prod_{j=0}^{n-1}(2j+1)\prod_{k=1}^{m}(r-m-n+2k)}R^{(r+1)}_{m,n-1},\\
&D\begin{bmatrix} m & m+1\\ 1 & m+n+1\end{bmatrix}=R^{(r-1)}_{m-1,n}.
\end{aligned}
\tag{4.23}
$$

It is easy to see that the bilinear relations (4.16) and (4.17) follow immediately from Jacobi's identity (4.18) by using Lemmas 4.5 and 4.6, respectively. We give the proof of Lemmas 4.5 and 4.6 in Appendix B. This completes the proof of Proposition 4.2 and thus our main result Theorem 1.2.

5 Discussion

As we mentioned in Section 2.3, the τ- cocycles $\phi_{k,l,m,n}$ are polynomials in α_i and f_i and admit a determinant expression in terms of a generalized Jacobi-Trudi formula [21]. When specialized to the seed solution (3.1), we obtain a determinant formula for $\phi_{0,0,m,n} = \phi_{m,n}$, which are given as follows.

Proposition 5.1 *Let $g_k^{(l)}$ $(k, l \in \mathbb{Z})$ be functions defined by*

$$\begin{aligned} g_{2k}^{(2l)} &= \frac{(-1)^k}{\xi_k} L_k^{(2s-1)}(t/2), \quad g_{2k+1}^{(2l)} = \frac{\sqrt{t}}{2} \frac{(-1)^k}{\xi_{k+1}^\dagger} L_k^{(2s^\dagger)}(t/2), \\ g_{2k}^{(2l+1)} &= \frac{(-1)^k}{\xi_k^\dagger} L_k^{(2s^\dagger-1)}(t/2), \quad g_{2k+1}^{(2l+1)} = \frac{\sqrt{t}}{2} \frac{(-1)^k}{\xi_{k+1}} L_k^{(2s)}(t/2), \end{aligned} \tag{5.1}$$

with

$$\xi_k = \xi_k(s) = \prod_{j=1}^{k} \left(s + \frac{j-1}{2} \right), \quad \xi_k^\dagger = \xi_k(s^\dagger), \tag{5.2}$$

where $L_k^{(r)}(x)$ are the Laguerre polynomials and $s^\dagger = 1/2 - s$. Then, $\phi_{m,n}$ under the specialization (3.1) are given by

$$\phi_{m,n} = N_{m,n} \det \left(g_{\lambda_j - j + i}^{(m+n+1-i)} \right)_{i,j=1}^{m+n}, \tag{5.3}$$

where the partition λ and the normalization factor $N_{m,n}$ are given by

$$\lambda = \begin{cases} (3m-n-1, 3m-n-4, \cdots, 2n+5, 2n+2, 2n, 2n, \cdots, 4, 4, 2, 2), & (m > n), \\ (3n-m, 3n-m-3, \cdots, 2m+3, 2m, 2m, \cdots, 4, 4, 2, 2), & (m \le n), \end{cases} \tag{5.4}$$

and

$$N_{m,n} = \begin{cases} (-1)^{n(n+1)/2} c_m d_n \prod_{k=1}^{n} \hat{\zeta}_k \prod_{k=1}^{m} \zeta_k^\dagger \prod_{k=1}^{m-n-1} \hat{\zeta}_k^\dagger & (m > n), \\ (-1)^{n(n+1)/2} c_m d_n \prod_{k=1}^{n} \hat{\zeta}_k \prod_{k=1}^{m} \zeta_k^\dagger \prod_{k=1}^{n-m} \zeta_k, & (m \le n), \end{cases} \tag{5.5}$$

with

$$\zeta_k = \prod_{j=1}^{k} (s + j - 1), \quad \hat{\zeta}_k = \prod_{j=1}^{k} \left(s + \frac{2j-1}{2} \right), \tag{5.6}$$

respectively.

This gives a different expression for the rational solutions discussed in this paper. Studying the relationship between this formula and our result might be an interesting problem.

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A Determinant Formula for the Umemura Polynomials

In [9], Noumi and Yamada gave a determinant formula of Jacobi-Trudi type for the Umemura polynomials in terms of 2-reduced Schur functions. In this appendix, we give a brief review on this determinant formula, and show that it is recovered as a special case of our formula.

We normalize the polynomials $U_{m,n}$ in Section 3 as

$$U_{m,n} = 2^{m(m+1)/2+n(n+1)/2} T_{m,n}. \tag{A.1}$$

Then we find that the functions $T_n = T_{0,n}(t,s)$ are monic polynomials generated by the Toda equation

$$T_{n+1}T_{n-1} = t\left[\left(\frac{d^2T_n}{dt^2}\right)T_n - \left(\frac{dT_n}{dt}\right)^2\right] + \frac{dT_n}{dt}T_n + \left(\frac{t}{8} - \frac{2s+1+3n}{4}\right)T_n^2, \tag{A.2}$$

with initial conditions $T_{-1} = T_0 = 1$. The polynomials T_n are called Umemura polynomials. It is easy to see that we have

$$T_{-1,n}(t,s) = T_{0,n}(t,s+1/2). \tag{A.3}$$

Introducing $\hat{T}_n(t,s)$ by $\hat{T}_n(t,s) = T_{-n}(t,s)$, we have the following proposition.

Proposition A.1 *Let $\hat{T}_n = \hat{T}_n(t,s)$ be a sequence of polynomials in t and s defined through the Toda equation*

$$\hat{T}_{n+1}\hat{T}_{n-1} = t\left[\left(\frac{d^2\hat{T}_n}{dt^2}\right)\hat{T}_n - \left(\frac{d\hat{T}_n}{dt}\right)^2\right] + \frac{d\hat{T}_n}{dt}\hat{T}_n + \left[\frac{t}{8} - \frac{1}{2}\left(s+\frac{1}{2}\right) + \frac{3}{4}n\right]\hat{T}_n^2, \tag{A.4}$$

with initial conditions $\hat{T}_0 = \hat{T}_1 = 1$. Then, the rational function

$$y = -\frac{\hat{T}_{n+1}(t,s)\hat{T}_n(t,s+1/2)}{\hat{T}_{n+1}(t,s+1)\hat{T}_n(t,s-1/2)}, \tag{A.5}$$

solves $P_{\rm V}$ with the parameters

$$\kappa_\infty = s, \quad \kappa_0 = s-n, \quad \theta = -n-1. \tag{A.6}$$

The explicit formula for $\hat{T}_n$ was given by Noumi and Yamada, which is expressed in terms of the 2-reduced Schur functions.

Proposition A.2 *Let $S_n = S_n(t_1, t_2, \cdots)$ for $n \geq 0$ be the Schur function associated with a partition $\lambda = (n, n-1, \cdots, 2, 1)$. Then, we have*

$$\hat{T}_{n+1}(t,s) = N_n S_n \tag{A.7}$$

with

$$\begin{gathered} N_n = 2^{-n(n+1)}(2n-1)!!(2n-3)!!\cdots 3!!1!!, \\ t_j = \frac{t}{2} + \frac{-2s+n+1}{j}. \end{gathered} \tag{A.8}$$

It is easy to verify that Proposition A.2 is recovered by putting $m=0$ for the solutions (1.8) with (1.9) in Theorem 1.2.

B Proof of Lemmas 4.5 and 4.6

We first note that the following contiguity relations hold by definition,

$$p_k^{(r)} - p_{k-1}^{(r)} = p_k^{(r-1)}, \quad q_k^{(r)} - q_{k-1}^{(r)} = q_k^{(r-1)}, \tag{B.1}$$

and

$$(k+1)p_{k+1}^{(r)} = rp_k^{(r+1)} - xp_k^{(r+2)}, \quad (k+1)q_{k+1}^{(r)} = rq_k^{(r+1)} + xq_k^{(r+2)}. \tag{B.2}$$

Let us prove Lemma 4.5. Noticing that $p_1^{(r)} = 1$ and $p_k^{(r)} = 0$ for $k < 0$, we see that $R_{m,n}^{(r)}$ can be rewritten as

$$R_{m,n}^{(r)} = \begin{vmatrix} q_1^{(r)} & q_0^{(r)} & \cdots & q_{-m-n+3}^{(r)} & q_{-m-n+2}^{(r)} & q_{-m-n+1}^{(r)} \\ q_3^{(r)} & q_2^{(r)} & \cdots & q_{-m-n+5}^{(r)} & q_{-m-n+4}^{(r)} & q_{-m-n+3}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ q_{2m-1}^{(r)} & q_{2m-2}^{(r)} & \cdots & q_{m-n+1}^{(r)} & q_{m-n}^{(r)} & q_{m-n-1}^{(r)} \\ p_{n-m}^{(r)} & p_{n-m+1}^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} & p_{2n}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p_{-n-m+4}^{(r)} & p_{-n-m+5}^{(r)} & \cdots & p_2^{(r)} & p_3^{(r)} & p_4^{(r)} \\ p_{-n-m+2}^{(r)} & p_{-n-m+3}^{(r)} & \cdots & p_0^{(r)} & p_1^{(r)} & p_2^{(r)} \\ p_{-n-m}^{(r)} & p_{-n-m+1}^{(r)} & \cdots & p_{-2}^{(r)} & p_{-1}^{(r)} & p_0^{(r)} \end{vmatrix}. \tag{B.3}$$

Subtracting the $(j-1)$-st column from the j-th column of $R_{m,n}^{(r+1)}$ for $(j = m+n, m+n-1, \cdots, 2)$ and using (B.1), we get

$$R_{m,n}^{(r+1)} = (-1)^m \begin{vmatrix} -q_1^{(r+1)} & q_1^{(r)} & \cdots & q_{-m-n+4}^{(r)} & q_{-m-n+3}^{(r)} \\ -q_3^{(r+1)} & q_3^{(r)} & \cdots & q_{-m-n+6}^{(r)} & q_{-m-n+5}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -q_{2m-1}^{(r+1)} & q_{2m-1}^{(r)} & \cdots & q_{m-n+2}^{(r)} & q_{m-n+1}^{(r)} \\ p_{n-m}^{(r+1)} & p_{n-m+1}^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{-n-m+4}^{(r+1)} & p_{-n-m+5}^{(r)} & \cdots & p_2^{(r)} & p_3^{(r)} \\ p_{-n-m+2}^{(r+1)} & p_{-n-m+3}^{(r)} & \cdots & p_0^{(r)} & p_1^{(r)} \end{vmatrix}. \tag{B.4}$$

From (B.3) and (B.4), we obtain Lemma 4.5.

We next prove Lemma 4.6. Subtracting the $(i+1)$-st column from the i-th column of $R_{m,n}^{(r)}$ for $(i = 1, 2, \cdots, j,\ j = m+n-1, m+n-2, \cdots, 1)$ and using (B.1), we get

$$R_{m,n}^{(r)} = \begin{vmatrix} q_1^{(r-m-n+1)} & q_0^{(r-m-n+2)} & \cdots & q_{-m-n+3}^{(r-1)} & q_{-m-n+2}^{(r)} \\ q_3^{(r-m-n+1)} & q_2^{(r-m-n+2)} & \cdots & q_{-m-n+5}^{(r-1)} & q_{-m-n+4}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1}^{(r-m-n+1)} & q_{2m-2}^{(r-m-n+2)} & \cdots & q_{m-n+1}^{(r-1)} & q_{m-n}^{(r)} \\ (-1)^{m+n-1}p_{2n-1}^{(r-m-n+1)} & (-1)^{m+n-2}p_{2n-1}^{(r-m-n+2)} & \cdots & (-1)^1 p_{2n-1}^{(r-1)} & (-1)^0 p_{2n-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m+n-1}p_3^{(r-m-n+1)} & (-1)^{m+n-2}p_3^{(r-m-n+2)} & \cdots & (-1)^1 p_3^{(r-1)} & (-1)^0 p_3^{(r)} \\ (-1)^{m+n-1}p_1^{(r-m-n+1)} & (-1)^{m+n-2}p_1^{(r-m-n+2)} & \cdots & (-1)^1 p_1^{(r-1)} & (-1)^0 p_1^{(r)} \end{vmatrix}. \tag{B.5}$$

Noticing that $p_0^{(r)}=1$ and $p_k^{(r)}=0$ for $k<0$, we see that $R_{m,n}^{(r)}$ can be rewritten as

$$R_{m,n}^{(r)}=\left|\begin{array}{ccccc} q_1^{(r-m-n)} & q_0^{(r-m-n+1)} & \cdots & q_{-m-n+2}^{(r-1)} & q_{-m-n+1}^{(r)} \\ q_3^{(r-m-n)} & q_2^{(r-m-n+1)} & \cdots & q_{-m-n+4}^{(r-1)} & q_{-m-n+3}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{2m-1}^{(r-m-n)} & q_{2m-2}^{(r-m-n+1)} & \cdots & q_{m-n}^{(r-1)} & q_{m-n-1}^{(r)} \\ (-1)^{m+n}p_{2n}^{(r-m-n)} & (-1)^{m+n-1}p_{2n}^{(r-m-n+1)} & \cdots & (-1)^1 p_{2n}^{(r-1)} & (-1)^0 p_{2n}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m+n}p_2^{(r-m-n)} & (-1)^{m+n-1}p_2^{(r-m-n+1)} & \cdots & (-1)^1 p_2^{(r-1)} & (-1)^0 p_2^{(r)} \\ (-1)^{m+n}p_0^{(r-m-n)} & (-1)^{m+n-1}p_0^{(r-m-n+1)} & \cdots & (-1)^1 p_0^{(r-1)} & (-1)^0 p_0^{(r)} \end{array}\right|. \tag{B.6}$$

We add the j-th column multiplied by $(r-2+m+n-j)/x$ to the $(j+1)$-st column of (B.6) for $(j=m+n,m+n-1,\cdots,1)$. Then using (B.1) and (B.2), we obtain

$$\begin{aligned} R_{m,n}^{(r)} \;=\; & \prod_{j=0}^{n}(2j+1)\prod_{k=1}^{m}(r-m-n+2k-2)x^{-(m+n)} \\ & \times \left|\begin{array}{ccccc} \tilde{q}_1^{(r-m-n)} & q_1^{(r-m-n-1)} & \cdots & q_{-m-n+3}^{(r-3)} & q_{-m-n+2}^{(r-2)} \\ \tilde{q}_3^{(r-m-n)} & q_3^{(r-m-n-1)} & \cdots & q_{-m-n+5}^{(r-3)} & q_{-m-n+4}^{(r-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{q}_{2m-1}^{(r-m-n)} & q_{2m-1}^{(r-m-n-1)} & \cdots & q_{m-n+1}^{(r-3)} & q_{m-n}^{(r-2)} \\ (-1)^{m+n}\hat{p}_{2n}^{(r-m-n)} & (-1)^{m+n}p_{2n+1}^{(r-m-n-1)} & \cdots & (-1)^2 p_{2n+1}^{(r-3)} & (-1)^1 p_{2n+1}^{(r-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m+n}\hat{p}_2^{(r-m-n)} & (-1)^{m+n}p_3^{(r-m-n-1)} & \cdots & (-1)^2 p_3^{(r-3)} & (-1)^1 p_3^{(r-2)} \\ (-1)^{m+n}\hat{p}_0^{(r-m-n)} & (-1)^{m+n}p_1^{(r-m-n-1)} & \cdots & (-1)^2 p_1^{(r-3)} & (-1)^1 p_1^{(r-2)} \end{array}\right|. \end{aligned} \tag{B.7}$$

Lemma 4.6 follows from (B.5),(B.6) and (B.7).

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